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$P_1P_2P_3$ (irrespective of which is the unknown), determining λ from the data (if it is not given explicitly), and then applying the relation. I have found that the complete elimination of the word "formula," in terms of which the student likes to think, and the substitution of the idea that the equation is a statement of a relation between various quantities is very helpful in all such problems.

A similar situation is met when dealing with the angle between two lines. The student can use the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

to find $\tan\theta$, but the problem of finding a line making a given angle with a given line is more difficult. This trouble, like the previous one, is avoided by impressing the fact that $(1 + m_1m_2)\tan\theta = m_1 - m_2$ is a relation between three quantities, m_1, m_2, and $\tan\theta$, any two of which being known the relation enables us to determine the third. It is not a mere change of terminology which is used for removing the difficulty; we are giving the student a better conception of the word *equation.*

THE GRAPH OF $F(X)$ FOR COMPLEX NUMBERS.

By A. F. FRUMVELLER, Marquette University.

Teachers of analytic geometry are often asked why the complex roots of $f(x)$ cannot be diagrammed. The answer usually given, that real points of the graph lie in the plane of the blackboard, while complex points are in front of this plane or behind it, is true enough; but it leaves the student unsatisfied. He is not able to carry out the idea for himself; and the teacher may not have thought of working it out, nor can he lay his hand on any systematic treatment of the matter. Yet the topic is important, not to say interesting, since several lines of thought here converge: a rather full development with ample illustrations may consequently be welcomed. The purposes of a mathematical club, or undergraduate seminar, would be beautifully served by a handling of graphs along the following lines. Care has been taken with the notation: x invariably stands for $u + iv$; y stands for $U + iV$; complex constants are designated by Greek letters; thus, $\alpha = a + iA$, $\rho = r + iR$, etc. We restrict $f(x)$ to algebraic types merely for the sake of uniformity in the presentation.

The literature on this subject is so slight that the writer is able to mention only three sources to which the student may go for further information; these are:

1. Schultze's "Graphic Algebra" (The MacMillan Company, 80 pp.): methods are given for the construction of the complex roots of quadratics, cubics, and quartics, the purpose being merely to find these roots, and not to exhibit the complete graph.

2. Hamilton and Kettle's "Graphs and Imaginaries" (Longmans, Green and Company, 40 pp.): the quadratic alone is treated. Complex roots, imaginary

intersections of two circles, tangent-points of a tangent to a circle from a point within, are constructed most elegantly by means of the polar and an auxiliary conic called the "shadow-curve." The method is not available for general graphing.

3. Phillips and Beebe's "Graphic Algebra" (Henry Holt and Company, 157 pp.): the last 60 pages deal with the complete complex graph of the reduced quadratic, cubic, and quartic, but only when their coefficients are real. Triangular or 60°-paper is used, and the complex branches are plotted in the usual way, point by point, but with many skilful simplifications. The methods of the present paper are not employed, so that the student should by all means consult this book in connection with the following section. See also the article on "Imaginaries in Geometry" by E. W. Davis, in *Nebraska University Studies*, volume 10.

§ 1. X COMPLEX, Y REAL.

When x is real, its domain or field is the line OX, and real values y_k of the function $y = f(x)$ are represented by perpendicular ordinates erected at x_k on OX. If now x belongs to the number-system $u + iv$, its field is the plane uov; and just as before, we associate with every point (u, v) of the plane a real perpendicular ordinate y. As x ranges over its plane, the values of $y = f(u + iv)$ will sometimes be real, sometimes complex; the real ordinates will be found standing on certain curves $u = \varphi(v)$, whose equations can at once be found. For example:

(*a*) $y = mx + n$. Since $y = (mu + n) + imv$, real ordinates occur only when $v = 0$, *i. e.*, they stand on the axis of reals, and there are none anywhere else in the plane. The plane uoy is of course identical with the plane YOX of ordinary analytic geometry—a fact to be remembered throughout.

(*b*) $y = kx^2 + mx + n$. When $x = u + iv$, we find

$$y = (ku^2 - kv^2 + mu + n) + iv(2ku + m).$$

If y is to be real, either $u + (m/2k) = 0$, or $v = 0$, so that all real ordinates stand on two straight lines in the x-plane; namely the axis of reals, and a perpendicular to it at $[-(m/2k), 0]$. When $v = 0$, we have $y_1 = ku^2 + mu + n$ as the equation of that branch of the graph lying in the plane uoy; when $u = -(m/2k)$, $y_2 = [n - (m^2/4k)] - kv^2$ is the other branch, lying in a plane parallel to voy. The complete locus consists, therefore, of two parabolic branches of opposite curvature in perpendicular planes with a common vertex-ordinate at $[-(m/2k), 0]$, whose length is $n - (m^2/4k)$. One branch never meets the x-plane; the other pierces it in two points, whose coördinates are precisely the two complex, or the two real, root-values of $f(x)$, according as the discriminant is minus or plus.

To make this clearer, take $y = x^2 - 3x + 5$ (Fig. 1).[1] On $v = 0$, and $u = \frac{3}{2}$, stand the real y's, defining the curves $y_1 = u^2 - 3u + 5$, and $y_2 = \frac{11}{4} - v^2$. This latter pierces the x-plane at $u = \frac{3}{2}$, $v = \pm \sqrt{11}/2$,

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[1] The drawings were made by Mr. Andrew Keiding, senior student at Marquette University.

i. e., at $x = \frac{3}{2} \pm (i/2)\sqrt{11}$, or at $x = OQ + iRQ$. In passing, we observe a curious relation between TQ and RQ. The minimum ordinate of $y = ku^2 + mu + n$, standing at $u = -(m/2k)$, is $TQ = (4kn - m^2)/4k = k \cdot RQ^2$; hence we have $x = OQ \pm (i/\sqrt{k})\sqrt{TQ}$.

(*c*) In the cubic $y = x^3$, the real y's stand on the 60° lines, $v = 0$, and $v = \pm u\sqrt{3}$. More generally, let $y = x^3 + mx + n$. When $x = u + iv$, we have $y = (u^3 - 3uv^2 + mu + n) + iv(3u^2 - v^2 + m)$. The real y's now stand on $v = 0$, and $v^2 - 3u^2 = m$, an hyperbola, and its axis, major or conjugate, according to the sign of m. The path-equations are equations of condition.

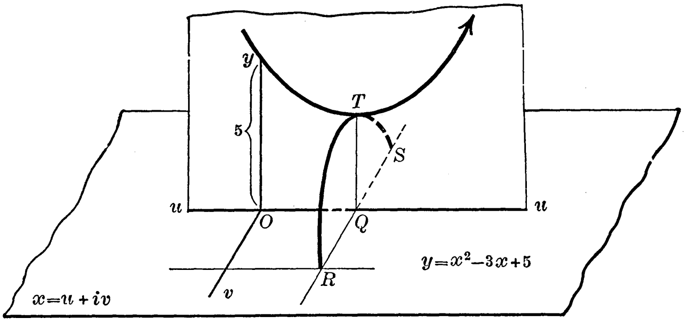

FIG. 1.

If we combine them with the original y-equation so as to eliminate v, we get as length-formulas for y, $y_1 = u^3 + mu + n$, and $y_2 = -8u^3 - 2mu + n$. These are the equations of the graphs that would be obtained by projecting the real ordinates from infinity parallel to ov onto the plane background uoy. (Similarly for the plane voy, by eliminating u.) The points on the hyperbola at which $y = 0$ give the complex roots of $x^3 + mx + n = 0$; they may be found algebraically by putting $y = f_3(u, v) + iF_3(u, v) = 0$, and solving $f_3 = 0$ and $F_3 = 0$ as simultaneous. Note that the plane uov contains only three isolated points; if $y = \sum_0^n a_k x^k = f_n(u, v) + iF_n(u, v)$, there would be n isolated points, since there exist, for every y, exactly n values $x_k = u_k + iv_k$ of x.

Nothing prevents our equations from having *complex coefficients*. Let $\alpha_1 x + \alpha_0 = y$. Then $y = (a_1 u - A_1 v + a_0) + i(a_1 v + A_1 u + A_0)$, where the real y's stand on $a_1 v + A_1 u + A_0 = 0$, which pierces the plane of reals in one point; hence the graph in the plane uoy is merely an isolated point. The quadratic, $(a_2 + iA_2)x^2 + (a_1 + iA_1)x + (a_0 + iA_0) = y$, has real y's on the curve $A_2(u^2 - v^2) + 2a_2 uv + a_1 v + A_1 u + A_0 = 0$; if $v = 0$, the only points on the real axis ou, at which ordinates can stand, are given by $A_2 u^2 + A_1 u + A_0 = 0$ (whose roots are not necessarily real) and the length of these ordinates is furnished by $a_2 u^2 + a_1 u + a_0 = 0$.

In general, if $y = \sum_0^n \alpha_k x^k$, and $v = 0$, we have $\sum_0^n A_k u^k = 0$, which gives, at most, n points on ou, the real axis; combining this with the conclusion of the previous paragraph, the result may be stated as follows:

Let $y = \sum_0^n \alpha_k x^k$. When x is $\left\{\begin{matrix}\text{complex}\\ \text{real}\end{matrix}\right\}$ and the coefficients $\left\{\begin{matrix}\text{real}\\ \text{complex}\end{matrix}\right\}$, the locus in the $\left\{\begin{matrix}\text{complex plane } uov\\ \text{real plane } uoy\end{matrix}\right\}$ consists of $\left\{\begin{matrix}n \text{ isolated points.}\\ k \text{ isolated points, } (k \leqq n)\end{matrix}\right\}$.

The condition for a real root x_r in $\sum_0^n \alpha_k x^k = 0$ is found by eliminating u from $\sum_0^n A_k u^k = 0$, and $\sum_0^n a_k u^k = 0$ (cf. Hardy, "Pure Mathematics," p. 88). The actual finding of all the real and complex roots of this equation is accomplished by solving $f_n(u, v) = 0$, and $F_n(u, v) = 0$ as simultaneous, either algebraically or graphically. The reader is advised to try the graphic solution in some simple case, such as $ix^2 + 8 = 0$, for instance. (Compare this method with that of Schultze, "Graphic Algebra," pp. 36, 51, 63.)

Implicit functions are dealt with as above, but require a little more care. Take $y^2 = x^3 = (u^3 - 3uv^2) + i(3u^2v - v^3)$; y is real when $v = 0$, and when $3u^2 - v^2 = 0$. This last gives $y^2 = -8u^3$, so that u must be negative for this part of the locus. But above all, the student should carefully construct and visualize the graph of $x^2 + y^2 - r^2 = 0$. The case of $x^2 + y^2 + r^2 = 0$ gives $y^2 = v^2 - (r^2 + u^2)$, with the condition $uv = 0$: the branch $y_1{}^2 = -(r^2 + u^2)$ in the plane uoy is destitute of real ordinates, but in the plane voy we have the real hyperbola $y_2{}^2 = v^2 - r^2$. The locus $x^2 + y^2 = i$ has no real ordinates at all. The student should carefully draw the graph of $y^2 = x^2(x - 2)$, a curve with an isolated point at the origin. Other types suitable for graphic study are furnished by $x, x^2, x^3 = f(y)$; and by $y, y^2, y^3 = \prod_0^n (x - \alpha_k)$, or $1/\sum_0^n \alpha_k x^k$, for $n = 1, 2, 3$, employing in each equation first real, then imaginary, then complex numerical coefficients (cf. Phillips and Beebe, "Graphic Algebra," Chap. V, for a very complete treatment of the quadratic, cubic, and quartic, when the coefficients are real; many graphs are given in illustration).

§ 2. X REAL, Y COMPLEX.

At first sight, this section might seem superfluous, since on merely interchanging x and y, we come back to the preceding case. For the purpose we have in view, however, such an inversion is not allowable; we are making a clearcut distinction between the dependent and the independent variable, and while the latter ranges over its domain, the function is to be represented by an ordinate at right angles to this domain. The carrying out of this idea demands that we shall assign to x and to y, throughout this paper, an invariable meaning; the method of graphing to which we are thus led is of the highest theoretical value, since without it a clear understanding of the graph of $f(x)$, when both X and Y are complex, cannot be obtained.

The field of x is therefore a line XOX lying in space; y is of the form $U + iV$, or $y_r + y_i$. On a fixed "base-plane" UOV, perpendicular to OX, let OU, OV, be two directions at right angles for reference. These preparations having been made, let x run over its domain OX. At every point x_k we shall have a perpendicular ordinate y_k: if y_k is real, it stands out from OX parallel to OU—if imaginary, parallel to OV—if complex, at an intermediate angle depending on its amplitude, as given by the equation $y_k = f(x_k)$. The result, in general, will be a spiral curve wound around OX (Fig. 2).

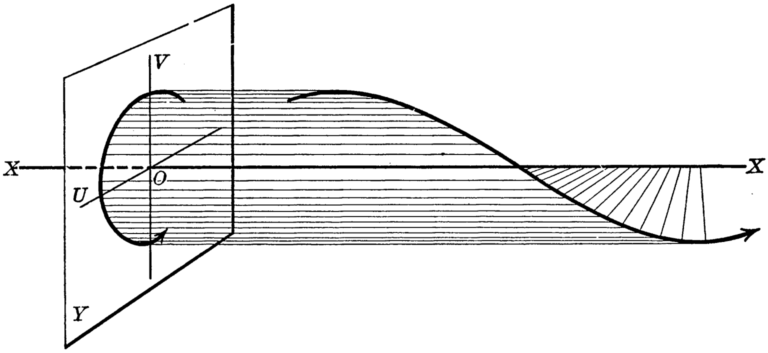

FIG. 2.

If now, we go to infinity and look back along OX, all these ordinates will be projected, unchanged in length and direction, onto the base-plane UOV, and will seem to radiate as vectors from the origin; their endpoints will map out a locus on UOV, say $U = F(V)$. This locus has its points P_k in one-to-one correspondence with the ordinates y_k, and consequently with the values x_k of the independent variable; it is therefore in a true, though slightly unusual sense, a graph of $y = f(x)$. In reality, it is the projection of the actual locus; but the UV-graph is more compact and easier to handle; and it furnishes all the information that the actual locus could give. It is evident that the function-symbol "f" acts as an operator, transforming the segment $x_m x_n$ of the x-domain into a

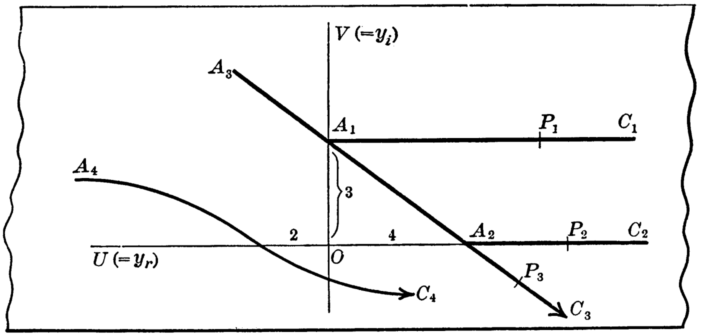

FIG. 3.

segment of the curve $U = F(V)$. This manner of graphing seems to be new, although it is only an extension of the usual process. Since x is thus lost to view, its path must be depicted on a separate diagram.

To have y complex, $f(x)$ must be of the form $\psi(x) + i\phi(x)$. If $\phi(x) = 0$, the graph is altogether in the plane UOX, and its projection is simply a segment of the real axis OU, since the ordinates all pile up on top of one another; similarly for $\psi(x) = 0$. The equation of the UV-curve in parametric form will be $U = \psi(x)$, $V = \phi(x)$. The endpoint of the vector $y_k = OP_k$ is (U_k, V_k), or $(\psi(x_k), \phi(x_k))$, and its angle is $\tan^{-1}(V_k/U_k)$. Let the reader now identify the following graphs (Fig. 3), and try to visualize the space-curves of which they are the projection:

A_1C_1,	Graph of $y = 4x^2 + 3i$,	Equation; $U = 4x^2$, $V = 3$,
A_2C_2,	" " $y = 4x^2 + 3$,	" $U = 4x^2 + 3$, $V = 0$,
A_3C_3,	" " $y = (4 - 3i)x^2 + 3i$,	" $3U + 4V - 12 = 0$.
A_4C_4,	" " $y = x^3 - 2 - ix$,	" $V^3 + U + 2 = 0$.

When the equation is in the form $y = \sum_0^n \alpha_k x^k$, we obtain $U = F(V)$ very readily by Sylvester's method of elimination: thus,

$$y = \sum_0^1 \alpha_k x^k \text{ gives the straight line} \begin{vmatrix} a_1 & a_0 - U \\ A_1 & A_0 - V \end{vmatrix} = 0;$$

$$y = \sum_0^2 \alpha_k x^k \text{ gives the conic,} \begin{vmatrix} a_2 & a_1 & a_0 - U & 0 \\ 0 & a_2 & a_1 & a_0 - U \\ A_2 & A_1 & A_0 - V & 0 \\ 0 & A_2 & A_1 & A_0 - V \end{vmatrix} = 0, \text{ and so on.}$$

Among the simpler types of $f(x)$ we may note the following:

$y = a_0 + iA_0$ is the point $(U, V) = (a_0, A_0)$. The actual locus is a line parallel to OX.

$y = (b + iB) \sum_0^n a_k x^k$ is the line $\dfrac{U}{b} = \dfrac{V}{B}$.

$y = (b + iB) \sum_1^n a_k x^k + (c + iC)$ is the line $\dfrac{U - c}{b} = \dfrac{V - C}{B}$; a special type of which is

$y = \alpha + (\beta - \alpha)x$, or $\dfrac{U - a}{b - a} = \dfrac{V - A}{B - A}$, the line joining $y = \alpha$, and $y = \beta$.

$y = \dfrac{\alpha + \beta x}{\gamma + \delta x}$, a circle. (Write the complex numbers in full, clear of fractions, etc., and finally eliminate x.)

$y = \gamma + \rho \frac{1 + ix}{1 - ix}$, a circle in explicit form; on developing this expression as above, we find $(U - c)^2 + (V - C)^2 - (r^2 + R^2) = 0$, so that $y = \gamma$ is the center, and $| \rho |$ the radius. (Hardy, "Pure Mathematics," p. 94.)

The reverse problem—given a graph $U = F(V)$, find $y = f(x)$—can be solved whenever U and V are expressible in parametric form. For example, $U^2 = 4V$ becomes $y = 2x + ix^2$: $U = mV + b$ gives $y = mx + b + ix$, or even $y = mx^n + b + ix^n$; $U^2 + V^2 - r^2 = 0$ becomes $y = r(\cos x + i \sin x)$, and so on. In a similar way, the value of x which makes $y = 0$, *i. e.*, which causes the graph $U = F(V)$ to pass through $(0, 0)$, will be found by putting $U = \psi(x) = 0$, $V = \phi(x) = 0$ simultaneously, and then obtaining the H.C.F. of these expressions.

Implicit functions like $y^2 = 4px$, $y^3 = x(x - 2)$, etc., are handled just as above. They have the peculiarity of furnishing both y_r and y_i when x as well as the coefficients are real. In Fig. 4 we have the complete locus of $x^2 + y^2 = 4$ where real ordinates lie in UOX, for $-2 < x < 2$. At $x = 2$ there is a sudden turn of 90°, but no discontinuity in the locus; the branch in VOX is $y = i\sqrt{x^2 - 4}$, or $y_i = V = \sqrt{x^2 - 4}$. The revolution of the figure about OX gives a two-sheeted hyperboloid with sphere, from which two mutually perpendicular sections through OX cut out the complete locus. The reader will note that tangents to $x^2 + y^2 = r^2$ may be drawn from points *within the circle*—a thing which at first sight would seem utterly impossible. Such tangents pass out of the two-space UOX into the next higher space. As a further exercise, one might draw the graph of $y^2 = x^2(x - 2)$ for the region $0 < x < 2$.

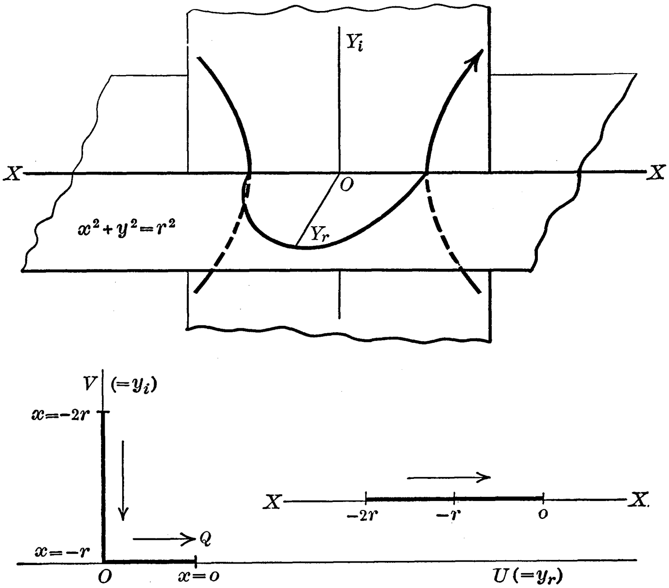

FIG. 4. Graphs of $x^2 + y^2 = r^2$.

The general case of $y^n = \psi(x) + i\phi(x)$ gives in a similar manner, at every point x_k, n different ordinates standing out from OX; a simple case for preliminary study would be $y^3 = x$, where two of the roots are complex, and one real. The arrangement of the ordinates can be followed by noting that

$$y = x^{1/3}(+1)^{1/3} = x^{1/3}\left[\cos\left(\frac{0^\circ + n \cdot 360^\circ}{3}\right) + i \sin\left(\frac{0^\circ + n \cdot 360^\circ}{3}\right)\right], \text{ etc.},$$

by Demoivre's theorem; on UOV, the graph consists of the U-axis, and two straight lines through (0, 0) making with it an angle of 60°. Each of these lines represents a branch of this three-valued function. To indicate this clearly, we may imagine that the plane UOV consists of three transparent, infinitely thin films, not connected with one another at the origin—each film depicting the march of one branch. For the circle, Fig. 4, the situation is different. Let $y = +\sqrt{r^2 - x^2}$ be the positive branch of the function; as x travels from $-\infty$ to $-r$, the point (U, V) travels down the axis OV from $+\infty$ to 0; and thence along OU to Q, while x moves on to zero. As x moves from zero to $+\infty$, (U, V) retraces its path, thereby producing an *overlapping;* to prevent this, conceive UOV as a double film, and cut these films along OU from Q to $+\infty$; then cross-connect the films along this cut, so that the point (U, V), on reaching Q, slides down onto the second film. It must remain on this sheet till x reaches $+\infty$; and in this way, a one-to-one correspondence of the x-path, and of its transformed path in UOV, is effected. A similar cut will go from $(-r, 0)$ to $-\infty$, for the branch $y = -\sqrt{r^2 - x^2}$. (When the plane OUV is thus regarded as composed of several sheets or films cross-connected suitably, it is called a "Riemann surface.") Graphs in UOV, composed of straight-line segments, as above, parallel to the axes, can be defined only by parametric equations.

Our present method of graphing affords a clear explanation of *imaginary intersections* of curves. Take the circles $x^2 + y^2 = 25$, and $(x - 26)^2 + y^2 = 25$; they fail to meet in the plane YOX, and when solved together give $x = 13$, $y = \pm 12i$. Let us state the problem this way: "Where will these two loci meet?" Answer: "On the locus $x = 13$;" which is absolutely correct, according to Fig. 4, since $x = 13$ is a plane perpendicular to the axis OX, on which the two hyperbolic branches cross. The situation is like that in algebra, where the equation is more general than the verbal statement, and applies to conditions not explicitly foreseen. Our imaginary intersections are *actual geometric entities* in a higher space-domain,—they are answers to a wider question in a larger number-field.

In a similar way, the statement that the *polar is the chord of contact of tangents from the pole* is substantially true, even when the pole is within the locus. A circle, for example, sends out, from an inner point P on the X-axis, two actual tangents to the hyperbolic branch, and these have a chord of contact standing *perpendicular* to the axis of reals. A plane at right angles to OX through this chord leaves as its trace on the plane of the circle the line which we call the *polar* of P. The *power-axis* of two non-intersecting circles is accounted for, in the same way, by the plane through their common chord (*cf.* Hamilton and Kettle, "Graphs and Imaginaries," ch. 5).

As a final application of the above methods, let us graph on the diagram of Fig. 4 the two curves $y^2 + x^2 = r^2$, and $Y^2 + (ix)^2 = - r^2$. On the shelf y_rox, we find the loci $y_r^2 + x^2 = r^2$, and $Y_r^2 + (ix)^2 = - r^2$; on the vertical plane y_iox, we find $Y_i^2 + (ix)^2 = - r^2$, and $y_i^2 + x^2 = r^2$, in the order given. Now, our $y = f(x)$ is essentially a "circular" function, and $Y = f(ix)$ is "hyperbolic." When $r = 1$, it is customary to represent the ordinates of these two classes of functions by the symbols Sin and Sinh, respectively, so that $y = f(x) = \text{Sin}\ (x)$, and $Y = f(ix) = \text{Sinh}\ (ix)$. The distances from their feet to the origin along OX are called Cos and Cosh, in like manner. The branches of our two loci are evidently congruent; hence, $\text{Cosh}\ (ix) = \text{Cos}\ (x)$, and $\text{Sinh}\ (ix) = i\ \text{Sin}\ (x)$, —a new and purely geometrical proof of the fundamental formulas of hyperbolic trigonometry.

The reader will note that if the X-axis of the graphs treated in this § 2 be identified with the real X-axis of the graphs in § 1, the imaginary Y-branches that we have discussed will lie in the fourth dimension.

§ 3. X AND Y BOTH COMPLEX.

The field of x is now the plane uov. When x was linear, its real and imaginary ordinates were at right angles, and both stood perpendicular to OX at the point x_0. The complex intermediate vectors y_c were likewise perpendicular to OX at the same point. To carry out this idea, there must now stand on every point $x_0 = u_0 + iv_0$ a real ordinate y_r and an imaginary ordinate y_i, mutually at right angles, and both perpendicular to uov. We need consequently a 4-dimensional space for our graph, y_i pointing into the fourth dimension. The plane $y_rx_0y_i$ may be called the "absolutely" perpendicular plane at x_0; in it, radiating from x_0, lie the other complex ordinates y_c, all at right angles to the plane uov at x_0.

The easiest way to deal with this situation graphically is first to ignore one dimension, say y_i, and plot the locus in the 3-space y_ruv; then ignore y_r, and plot in the space y_iuv. A clear idea can thus be gotten of what is happening in these two directions. Real ordinates will stand on certain paths in uov, and imaginary ones likewise: thus, if $y = x^2 - 2x + 4 = (u^2 - y^2 + 2u + 4) + 2iv(u - 1) = U + iV$, real ordinates whose lengths are, respectively, $y_r = u^2 - 2u + 4$, and $y_r = 3 - v^2$, stand on the paths $v = 0$, $u = 1$. Imaginary ordinates stand on the hyperbola $u^2 - v^2 - 2u + 4 = 0$, and their lengths are given by $y_i = 2(uv - v)$. Combining the path-equation with the length-equation by eliminating u or v, we get the equation of the locus that results from projecting all the y_r's, or y_i's, onto the planes y_rou, y_rov, or y_iou, y_iov.

In this example, U and V were explicit, and the critical paths for x could be read off at first sight; but now, take the circle $x^2 + y^2 - \rho^2 = 0$. Separating the real and imaginary terms, we get (1) $u^2 + U^2 - r^2 - (v^2 + V^2 - R^2) = 0$, (2) $uv + UV - rR = 0$; no obvious x-paths present themselves. If however we assign to v the value $\psi(u)$, we can eliminate u from (1) and (2), and thus obtain an equation between U and V. To any assigned curve $\psi(u, v) = 0$, there corresponds a definite curve $\phi(U, V) = 0$ in UOV, the "absolutely" perpendicular plane. One locus is the "transformation" of the other. The "f" in $y = f(x)$ is therefore an operator which transforms an arc of the locus $\psi = 0$ in planes 1 and 2 of our 4-space, into a corresponding arc of the locus $\phi = 0$ in planes 3 and 4. Its action is analogous to that of a mirror with a bent surface: the same function ψ, when it is transmitted through the medium of $f_1, f_2, \cdots$, comes out on the plane UOV as $\phi_1, \phi_2, \cdots$. This process is often spoken of as the *transformation* or *depiction* of the plane uov on the plane UOV.

The manner in which this depiction is brought about is as follows: Let there be given a locus $\psi(u, v) = 0$ in the x-plane, with all its ordinates, as computed from $y = f(x)$, standing upon it. Now project the plane from ∞, parallel to itself and to the line ov, onto the axis of reals ou. This leaves the *length* and the *inclination* of the ordinates y_c unchanged, because they are all perpendicular to uov, and their mutual angles are in the plane $y_roy_i \equiv UOV$, which is perpendicular to the line ov; hence they now stand in proper length and position on the line ou. Now project this line, parallel to itself, onto the plane UOV; all the ordinates pile up against this plane and appear as vectors issuing from the origin, and their end-points map out a locus $\phi(U, V) = 0$, which is the graph of $y = f(x)$ with $x = u + iv$ as invisible parameter.

Since x has been shuffled out of sight by projecting its field uov into the point $(0, 0)$ of the Y-plane, two separate diagrams will be needed in plotting—one, to show the path of x in its own plane—the other, to show the position and length of the vector-ordinates in the plane of Y. We may summarize the relation of these two planes as follows for explicit functions:

Let $y = f(x)$, and $x = u + iv$. To any point (u, v) corresponds a point (U, V), where $U = f_1(u, v)$, $V = f_2(u, v)$, and $y = U + iV$. To the x-path $v = \psi(u)$ corresponds the y-path $U = f_1(u, \psi(u))$, $V = f_2(u, \psi(u))$, or $F(U, V) = 0$; to the y-path $V = \phi(U)$ corresponds the x-path $f_2(u, v) = \phi[f_1(u, v)]$; it being understood that $\psi(u)$, and $\phi(U)$ may be constants. A given ordinate, $y = a + ib$, stands on the points (u_m, v_m) in the x-plane, where u_m, v_m are the common solutions of $U = f_1(u, v) = a$, $V = f_2(u, v) = b$. When $a = b = 0$, we find the complex roots of $f(x) = 0$. For instance, if $x^2 + y^2 = i$, and $U = V = 0$, we get $u = v$, and $uv = 1/2$, which meet at $(\pm (\sqrt{2}/2), \pm (\sqrt{2}/2))$ in the plane uov, where $y = 0$. We have thus found the square roots of i. From $x^3 + y = i$, we get in like manner $(0, -1)$, $(\pm (\sqrt{3}/2), 1/2)$; hence $-i$, and $(i/2) \pm (\sqrt{3}/2)$ are cube roots of i.

For implicit functions $\chi(x, y) = 0$, the separation of real and imaginary terms will yield $\chi_1(u, v, U, V) = 0$, $\chi_2(u, v, U, V) = 0$. To find the locus in UOV corresponding to $v = \psi(u)$, we must regard u as a parameter, and eliminate it, as above. If this be impossible, we can plot by points, though this is hardly ever useful in practice. Now let the reader try all this out on some simple function like $y = x^2$, $y^2 = \alpha x$, or $y = \alpha x^2 + \beta x + \gamma$—allowing x (or y) to run along lines parallel to the axes, or through the origin; or along rectangular hyperbolas, etc. The following examples afford useful practice:

(1) $y = (x - \beta)/(x - \alpha)$, when y is fixed, is the equation of a circle in *uov*. Write in full, separate the reals and imaginaries, solve for U and V by determinants; then

$$\frac{U}{V} = \frac{u^2 + v^2 - (a + b)u - (A + B)v + (ab + AB)}{(A - B)u + (b - a)v + (aB - Ab)}.$$

Hence, for any constant values of U and V, *i. e.*, for any fixed ordinate $y_0 = U_0 + iV_0$, we have a circle in *uov* as the locus.

(2) $y = \alpha x$, when α is real, effects a *magnification;* when α is imaginary, a *magnification* and a *rotation of* 90°; when α is complex, change x and α to polar form, and use DeMoivre's Theorem.

(3) $y = 1/x$, in polar form, shows us an *inversion* on the unit circle, and a *mirroring* in the real axis. What UV-curves correspond to circles in *uov*, center at origin? To straight lines through (o, o)? To the circle $a(u^2 + v^2) + bu + cv + d = 0$? Solve these questions for the transforming function $y = (1 + x)/(1 - x)$.

(4) $y = (\alpha x + \beta)/(\gamma x + \delta)$ transforms all circles into circles. (Compare (1) above.) It gives a straight line in UOV when $u^2 + v^2 = 1$, and $|\alpha| = |\gamma|$.

(5) $y = x + \alpha$ is merely a parallel transformation.

(6) $\alpha x + \beta y + \gamma = 0$ is an "imaginary" straight line; only one pair of real values (x, y) lies on this line. When the coefficients are real, it has an infinite number of conjugate imaginary points. When will the join of two imaginary points (lines) be real?

(7) Given two simultaneous equations, say $x^2 + \beta_1 y + \gamma_1 = 0$, and $x^2 + \beta_2 y + \gamma_2 = 0$, we can eliminate x, *i. e.*, u and v, at one stroke, and get an equation in y alone; from which, by separating reals and imaginaries, we find U and V. Using these, we next find u and v. This is a generalization of the ordinary case.

Conclusion.

Having gone thus far, we are at the threshold of the "theory of functions," which has for its object the study of the transformations effected by the various types of the functional operator "f," on the equation $y = f(x)$. One function depicts the whole x-plane in a sector only of the y-plane—another on a horizontal band or strip—a third, in the interior of a circle—etc. What kind of transformation, for instance, is brought about by the functions log, sin, cosh, exp, cube-root? Or again, given a y-domain with prescribed boundaries, zeros, infinities, singularities—what form of function "f" would produce this effect? In dealing with these questions, graphs of the kind explained above are constantly used.

The chief points of interest to the student in our discussion will be the use of a uniform and fixed principle in graphing $f(x)$—and the manner in which we have bridged the gap between analytic geometry and function-theory, by way of the fourth dimension. There are many other ways of graphing imaginaries, no doubt; but the above is to be preferred, because it is a logical and systematic scheme resting at both ends on the approved processes.

AN INTRINSIC EQUATION SOLUTION OF A PROBLEM OF EULER.

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In his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne (1744), p. 64, Euler proposed and solved the following problem: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. This area is of course expressed by the definite integral $\int_{s_0}^{s_1} \frac{1}{2}\rho ds$, in which ρ is the radius of curvature. The ordinary method of solution is to express the integral in the form

$$\int_{x_0}^{x_1} \frac{(1+y'^2)^2}{2y''}\,dx, \tag{1}$$

and to find, by methods of the calculus of variations, the form of the curves that will give it a minimum value. (It is to be observed that y'', the second derivative of y with respect to x, occurs in the integrand. The problem is of particular interest on this account.) It is necessary to make some substitutions and transformations that are not altogether simple, before it is evident that the minimizing curves are cycloids.[1]

I wish to give here a shorter solution by obtaining the intrinsic equation of the extremals (*i. e.*, minimizing curves). The intrinsic equation of a curve is defined as the relation between s, the length of arc measured from a fixed point on it, and ϕ, the angle of deviation of the tangent at any point from the tangent at the fixed point taken as origin. This is in the sense of Whewell.[2]

If the area integral (1) is written in the form

$$\int_{s_0}^{s_1} \frac{ds}{2\kappa}. \tag{2}$$

where κ is the curvature, we have a special case of an integral studied by Radon,[3]

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[1] See, for example, Euler, *loc. cit.*, or the solution outlined by Bolza, *Vorlesungen über Variationsrechnung*, p. 152.

[2] See *Cambridge Philosophical Transactions*, Vols. 8 and 9; or Williamson, *Differential Calculus*, 9th edition, revised (1899), p. 304.

[3] Radon, "Über das Minimum des Integrals $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften mathematisch-naturwissenschaftliche Klasse*, Wien, Vol. 119 (1910), pp. 1257–1326.